Elliptic stability for stationary Schrödinger equations

by

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Part I/VI

Building equations

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NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

PART I. THE BUILDING OF EQUATIONS

I.1) The Yamabe equation :

Let (M, g) be a closed Riemannian manifold (compact without boundary) of dimension $n \geq 3$. Let $\tilde{g}$ be a conformal metric to g. By definition, $\tilde{g} = fg$, where $f : M \to \mathbb{R}$ is a smooth positive function. Writing $f = e^{2\varphi}$, for $\varphi \in C^\infty(M, \mathbb{R})$, we get that

$$\mathrm{Rm}_{\tilde{g}} = e^{2\varphi}\left(\mathrm{Rm}_g - g \circledast \left(\nabla^2\varphi - \nabla\varphi \otimes \nabla\varphi + \frac{1}{2}|\nabla\varphi|^2 g\right)\right) ,$$

where Rm_g stands for the Riemann curvature of g and $\mathrm{Rm}_{\tilde{g}}$ for the Riemann curvature of $\tilde{g}$. In this equation,

(1) $\nabla^2\varphi$ is the Hessian of φ, given in local coordinates by $(\nabla^2\varphi)_{ij} = \partial^2_{ij}\varphi - \Gamma^\alpha_{ij}\partial_\alpha\varphi$, where the Γ^k_{ij}'s are the Christoffel symbols of the Levi-Civita connection of g,

(2) $\otimes$ is the tensorial product, $(\nabla\varphi \otimes \nabla\varphi)_{ij} = \partial_i\varphi\partial_j\varphi$ in local coordinates,

(3) $\circledast$ is the Kulkarni-Nomizu product acting on two times symmetric covariant tensor fields by $(H \circledast K)_{ijkl} = H_{ik}K_{jl} + H_{jl}K_{ik} - H_{il}K_{jk} - H_{jk}K_{il}$.

Let $\mathrm{Rc}_g, \mathrm{Rc}_{\tilde{g}}$ be the Ricci curvatures of g and $\tilde{g}$. The Ricci curvatures are the trace of the Riemann curvature. In local coordinates the components R_{ij} of Rc_g are given by $R_{ij} = g^{\alpha\beta} R_{i\alpha j\beta}$, where the R_{ijkl} are the components of Rm_g and the g^{ij}'s are the components of the inverse matrix of the matrix with components g_{ij}. Similarly, $\tilde{R}_{ij} = \tilde{g}^{\alpha\beta} \tilde{R}_{i\alpha j\beta}$. From the equation in the preceeding slide we get that

$$\mathrm{Rc}_{\tilde{g}} = \mathrm{Rc}_g - (n-2)\nabla^2\varphi + (n-2)\nabla\varphi \otimes \nabla\varphi + \left(\Delta_g\varphi - (n-2)|\nabla\varphi|^2\right) g \ ,$$

where Δ_g is the Laplace-Beltrami operator, given in local coordinates by

$$\begin{aligned} \Delta_g u &= -g^{ij}(\nabla^2 u)_{ij} \\ &= -g^{ij}\left(\partial^2_{ij} u - \Gamma^k_{ij}\partial_k u\right) \ . \end{aligned}$$

Let S_g and $S_{\tilde{g}}$ be the scalar curvatures of g and $\tilde{g}$, the total traces of the corresponding Riemann curvatures. In local coordinates, $S_g = g^{ij} R_{ij}$ and $S_{\tilde{g}} = \tilde{g}^{ij}\tilde{R}_{ij}$. By the above equation, we get that

$$e^{2\varphi} S_{\tilde{g}} = S_g + 2(n-1)\Delta_g\varphi - (n-1)(n-2)|\nabla\varphi|^2 \ .$$

The equation is not very pleasant, but it can easily be transformed into a nicer equation. For this we just write that $e^{2\varphi} = u^{\frac{4}{n-2}}$, where $u \in C^\infty(M, \mathbb{R})$ is now required to be positive.

Then $\varphi = \frac{2}{n-2} \ln u$ and we compute

$$
\begin{aligned}
(\nabla\varphi)_i &= \frac{2\partial_i u}{(n-2)u} \quad \left(\text{and } |\nabla\varphi|^2 = \frac{4|\nabla u|^2}{(n-2)^2 u^2}\right) \ , \\
(\nabla^2\varphi)_{ij} &= \frac{2\partial^2_{ij} u}{(n-2)u} - \frac{2\partial_i u \partial_j u}{(n-2)u^2} \ , \\
\Delta_g \varphi &= \frac{2\Delta_g u}{(n-2)u} + \frac{2|\nabla u|^2}{(n-2)u^2} \ .
\end{aligned}
$$

In the preceding equation

$$
e^{2\varphi} S_{\tilde{g}} = S_g + 2(n-1)\Delta_g \varphi - (n-1)(n-2)|\nabla\varphi|^2 \ ,
$$

the terms in $|\nabla u|^2$ disappear, and the equation then writes as

$$
u^{\frac{4}{n-2}} S_{\tilde{g}} = S_g + \frac{4(n-1)}{(n-2)u} \Delta_g u \ .
$$

The Yamabe equation is the equation which corresponds to the problem of finding a conformal metric $\tilde{g}$ to a given metric g such that $S_{\tilde{g}} = \text{Constant}$. The difficult case is the focusing case for which $S_{\tilde{g}} = 1$.

Summarizing, the Yamabe equation comes from conformal geometry and the equation relating the curvatures of two conformal metrics. In our framework ($S_{\tilde{g}} = 1$), the Yamabe equation (up to a positive scale factor) reads as

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^\star - 1} \,, \tag{Y}$$

where $2^\star = \frac{2n}{n-2}$. Let H^1 be the Sobolev space defined as the completion of $C^\infty(M, \mathbb{R})$ w.r.t. $\| \cdot \|_{H^1}$ given by $\|u\|_{H^1}^2 = \int_M \left(|\nabla u|^2 + u^2 \right) dv_g$. It turns out that $2^\star$ is the critical Sobolev exponent for the embeddings H^1 into Lebesgue spaces. The Yamabe equation is Sobolev critical.

Hidehiko Yamabe
1923-1960

I.2) The conformal Laplacian :

The operator in the left-hand side of (Y) is the conformal Laplacian

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} S_g \ .$$

In our case (focusing case), L_g needs to be positive if we want to have positive solutions to the equation. We recall that a stationary Schrödinger operator $\Delta_g + h$ is positive if there exists $C > 0$ such that $\|u\|_{H^1}^2 \le C \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$ for all $u \in H^1$. Another remark is that L_g is conformally invariant.

Conformal invariance : if g and $\tilde{g} = \varphi^{4/(n-2)} g$ are two conformal metrics, then

$$L_{\tilde{g}} u = \varphi^{-\frac{n+2}{n-2}} L_g(\varphi u)$$

for all u which we can differentiate two times. The following proof is natural from the viewpoint of analysis. Essentially, the only thing which needs to be proved is that

$$\varphi^{\frac{4}{n-2}} \Delta_{\tilde{g}} u = \Delta_g u - \frac{2}{\varphi} (\nabla \varphi \nabla u)_g$$

for all u, where $(\nabla \varphi \nabla u)_g = g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta u$ is the scalar product w.r.t. g.

Let ψ be any smooth function. Integrating by parts,

$$\begin{aligned}
\int_M (\Delta_{\tilde g} u)\psi dv_{\tilde g} &= \int_M (\nabla u \nabla \psi)_{\tilde g} dv_{\tilde g} \\
&= \int_M (\nabla u \nabla \psi)_g \varphi^2 dv_g \\
&= \int_M \left(\varphi^2 \Delta_g u - 2\varphi(\nabla u \nabla \varphi)_g\right) \psi dv_g \\
&= \int_M \left(\Delta_g u - \frac{2}{\varphi}(\nabla u \nabla \varphi)_g\right) \psi \varphi^{-\frac{4}{n-2}} dv_{\tilde g}
\end{aligned}$$

Since ψ is arbitrary, we get what we wanted to prove, namely that

$$\varphi^{\frac{4}{n-2}} \Delta_{\tilde g} u = \Delta_g u - \frac{2}{\varphi}(\nabla \varphi \nabla u)_g \tag{1}$$

Also we have (by direct computation) that

$$\Delta_g(u\varphi) = u\Delta_g \varphi + \varphi \Delta_g u - 2(\nabla u \nabla \varphi)_g \ , \tag{2}$$

and we have seen that

$$\Delta_g \varphi + \frac{n-2}{4(n-1)} S_g \varphi = \frac{n-2}{4(n-1)} S_{\tilde g} \varphi^{\frac{n+2}{n-2}} \ . \tag{3}$$

Then,

$$
\begin{aligned}
L_g(u\varphi) &= \varphi\Delta_g u + u\left(\Delta_g\varphi + \frac{n-2}{4(n-1)}S_g\varphi\right) - 2(\nabla u\nabla\varphi)_g \qquad \text{by (2)}\\
&= \varphi^{\frac{n+2}{n-2}}\Delta_{\tilde{g}} u + u\left(\Delta_g\varphi + \frac{n-2}{4(n-1)}S_g\varphi\right) \qquad \text{by (1)}\\
&= \varphi^{\frac{n+2}{n-2}}\Delta_{\tilde{g}} u + \frac{n-2}{4(n-1)}S_{\tilde{g}} u\varphi^{\frac{n+2}{n-2}} \qquad \text{by (3)}\\
&= \varphi^{\frac{n+2}{n-2}} L_{\tilde{g}} u\ .
\end{aligned}
$$

This proves the conformal invariance of the conformal Laplacian.

I.3) The Klein-Gordon-Maxwell-Proca equation :

This is a construction in quantum field theory which provides a model for the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. The particle field interacts with the external field via the minimum coupling rule in a nonlinear Klein-Gordon equation.

The minimum coupling rule reads as

$$\partial_t \to \partial_t + iq\varphi \text{ and } \nabla \to \nabla - iqA,$$

where (φ, A) gauge potential representing the electromagnetic field, governed by the Maxwell-Proca Lagrangian. Consider the two Lagrangian densities (to simplify the presentation we choose the pure power nonlinearity to be critical) :

$$\mathcal{L}_{NKG}(\psi, \varphi, A)$$
$$= \frac{1}{2}\left|(\frac{\partial}{\partial t} + iq\varphi)\psi\right|^2 - \frac{1}{2}\left|(\nabla - iqA)\psi\right|^2 - \frac{m_0^2}{2}|\psi|^2 + \frac{1}{p}|\psi|^{2^\star}$$

and

$$\mathcal{L}_{MP}(\varphi, A) = \frac{1}{2}\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{2}|\nabla \times A|^2 + \frac{m_1^2}{2}|\varphi|^2 - \frac{m_1^2}{2}|A|^2 .$$

Here $\nabla \times = \star d$, $\star$ Hodge dual, d differentiation. Massive version of KGM theory. Here ψ matter field, m_0 its mass, q its charge, (A, φ) gauge potentials representing the electromagnetic vector field, m_1 is the Proca mass.

Consider the total action functional

$$\mathcal{S}(\psi, \varphi, A) = \int\int (\mathcal{L}_{NKG} + \mathcal{L}_{MP})\, dv_g\, dt\ .$$

Write ψ in polar form as $\psi(x,t) = u(x,t)e^{iS(x,t)}$ for $u \geq 0$ and $u, S : M \times \mathbb{R} \to \mathbb{R}$. Then the total action rewrites as

$$\begin{aligned}
\mathcal{S}(u, S, \varphi, A) &= \frac{1}{2}\int\int \left(\left(\frac{\partial u}{\partial t}\right)^2 - |\nabla u|^2 - m_0^2 u^2\right) dv_g\, dt \\
&+ \frac{1}{p}\int\int u^p\, dv_g\, dt \\
&+ \frac{1}{2}\int\int \left(\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - |\nabla S - qA|^2\right) u^2\, dv_g\, dt \\
&+ \frac{1}{2}\int\int \left(\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - |\nabla \times A|^2 + \frac{m_1^2}{2}|\varphi|^2 - \frac{m_1^2}{2}|A|^2\right) dv_g\, dt\ .
\end{aligned}$$

We can take the variation of $\mathcal{S}$ with respect to u, S, φ, and A. For instance, if we let ω_g be the volume form of (M, g), then

$$\frac{1}{2}\left(\frac{d}{dA}\int |\nabla \times A|^2\right).(B) = \int (\star dA, \star dB)\,\omega_g \qquad (\text{quadratic} + \nabla\times = \star d)$$

$$= (-1)^{n-1}\int (\star dA, (\star d \star)\star B)\,\omega_g \qquad (\star\star = (-1)^{n-1} \text{ in } \Lambda^1)$$

$$= \int (\star dA, \delta \star B)\,\omega_g \qquad (\delta = (-1)^{n-1} \star d \star \text{ in } \Lambda^{n-1})$$

$$= \int (d \star dA, \star B)\,\omega_g \qquad (\text{Stokes formula})$$

$$= \int (\star\delta dA, \star B)\,\omega_g \qquad (d\star = \star\delta \text{ in } \Lambda^2)$$

$$= \int (\star\delta dA) \wedge (\star \star B) \qquad (\text{since } \alpha \wedge (\star\beta) = (\alpha, \beta)\omega_g \text{ in } \Lambda^p)$$

$$= (-1)^{n-1}\int (\star\delta dA) \wedge B \qquad (\star\star = (-1)^{n-1} \text{ in } \Lambda^1)$$

$$= \int (\delta dA, B)\,\omega_g \qquad (\alpha \wedge \beta = (-1)^{n-1}\beta \wedge \alpha \text{ for } \alpha \in \Lambda^{n-1}, \beta \in \Lambda^1)$$

In particular,

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B)=\int\left(\overline{\Delta}_g A,B\right)$$

for all B, where $\overline{\Delta}_g=\delta d$, δ codifferential. When $n=3$, $\overline{\Delta}_g=\nabla\times\nabla\times$. Taking the variation of

$$\mathcal{S}(\psi,\varphi,A)=\int\int\left(\mathcal{L}_{NKG}+\mathcal{L}_{MP}\right)dv_g dt\ .$$

with respect to u, S, φ, and A, we then get four equations which are written as

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}+\Delta_g u+m_0^2 u=u^{2^\star-1}+\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)^2-|\nabla S-qA|^2\right)u\\ \frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2\right)-\nabla.\left(\left(\nabla S-qA\right)u^2\right)=0\\ -\nabla.\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2\varphi+q\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2=0\\ \overline{\Delta}_g A+\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2 A=q\left(\nabla S-qA\right)u^2\ .\end{cases}\qquad (KGMP_f)$$

This is the nonlinear Klein-Gordon-Maxwell-Proca system. As $m_1\to 0$ (or letting $m_1=0$), the nonlinear KGMP system reduces to the nonlinear Klein-Gordon-Maxwell system.

Why we do refer to Maxwell-Proca : Assume $n = 3$. Let the electric field E, the magnetic induction H, the charge density ρ, and the current density J be given by

$$\begin{aligned} E &= -\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) , \\ H &= \nabla \times A , \\ \rho &= -\left(\frac{\partial S}{\partial t} + q\varphi\right) qu^2 , \\ J &= (\nabla S - qA)\, qu^2 . \end{aligned}$$

The two last equations in $(KGMP_f)$ give rise to the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J.

We get for free the second pair of the Maxwell-Proca equations. By Maxwell-Proca, we mean Maxwell equations in Proca form. To quote Louis de Broglie : these will be "des équations du type classique de Maxwell complétées par de petits termes contenant la masse propre".

In other words the two last equations in the $(KGMP_f)$-system can be rewritten as

$$\begin{aligned}
&\nabla.E = \rho - m_1^2\varphi \ , \\
&\nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ , \\
&\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla.H = 0 \ .
\end{aligned}$$

The first equation in the $(KGMP_f)$-system is the nonlinear Klein-Gordon matter equation. Namely

$$\frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{2^\star - 1} + \frac{\rho^2 - |J|^2}{q^2 u^3} \ .$$

The second equation in the $(KGMP_f)$-system is the charge continuity equation $\frac{\partial \rho}{\partial t} + \nabla.J = 0$, which is equivalent to the Lorentz condition

$$\nabla.A + \frac{\partial \varphi}{\partial t} = 0 \ .$$

The $(KGMP_f)$-system is equivalent to this system of 6 equations.

The equivalence between the charge continuity equation and the Lorentz condition involves only basic computations (and uses the condition $m_1 \neq 0$). The Maxwell-Proca equations are written as

$$\nabla.E = \rho - m_1^2\varphi \ , \quad \nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ ,$$
$$\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla.H = 0 \ .$$

The charge continuity equation states that $\frac{\partial \rho}{\partial t} + \nabla.J = 0$. Taking the derivation of the first Maxwell equation with respect to time, and the divergence of the second equation,

$$\frac{\partial \rho}{\partial t} + \nabla.J = \nabla.\frac{\partial E}{\partial t} + m_1^2\frac{\partial \varphi}{\partial t} + \nabla.(\nabla \times H) - \nabla.\frac{\partial E}{\partial t} + m_1^2\nabla.A$$
$$= m_1^2\left(\nabla.A + \frac{\partial \varphi}{\partial t}\right)$$

since $\nabla.(\nabla \times H) = \delta(\star d)H$, $\delta = \star^{-1}d\star$ in Λ^1, $\star\star = 1$ in Λ^2, and $d^2 = 0$ so that $\nabla.(\nabla \times H) = 0$. The condition $m_1 \neq 0$ breaks the gauge invariance and enforces the Lorentz gauge.

A short physics break : The Maxwell equations in Proca form are

$$\nabla.E = \rho - m_1^2\varphi \ ,$$
$$\nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ ,$$
$$\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla.H = 0 \ .$$

They reduce to the Maxwell equations as $m_1 \to 0$. Proca (1936) was using the Lorentz formalism. Under this form, referred to as the "modern format", the equations appeared for the first time in a paper by Schrödinger : "*The earth's and the sun's permanent magnetic fields in the unitary field theory*" (1943). These equations have been discussed by several physicists including, in addition to Proca and Schrödinger, people like De Broglie, Pauli , Yukawa, and Stueckleberg... The whole point in these theories is that m_1 is nothing but than the mass of the photon : we are talking about a theory where photons have a mass.

[1] G.T.Gillies, J.Luo, L.C.Tu, The mass of the photon, Report on Progress in Physics, 68, 2005, 77–130.
[2] A.S.Goldhaber, M.M.Nieto, Photon and Graviton mass limits, Reviews of Modern Physics, 82, 2010, 939–979.

Alexandru Proca
1897-1955

Louis de Broglie (1950) : A partir de 1934, l'auteur du présent article a développé une forme nouvelle de la théorie quantique du champ électromagnétique qu'il a appelé "la Mécanique ondulatoire du photon" et qui présentait à ses yeux l'avantage de faire plus clairement rentrer la théorie quantique des champs dans le cadre général de la Mécanique ondulatoire des particules à spin. Dans cette théorie, qui a été exposée dans plusieurs Ouvrages, il a été attribué au photon une masse propre extrêmement petite, mais non nulle, et nous avons été ainsi conduit dès 1934 à prendre comme équations de la particule de spin 1 des équations qui, mises sous forme vectorielle, sont des équations du type classique de Maxwell complétées par de petits termes contenant la masse propre. Des équations de même forme ont été ensuite proposées, en 1936, par M. Alexandre Proca, et on leur donne aujourd'hui dans la théorie du méson le nom d'équations de Proca. En somme ces équations sont les équations générales des particules de spin 1.

Back to the $(KGMP_f)$**-system - The reduced form :** Return to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{2^\star - 1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla . \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla . \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 A = q (\nabla S - qA) u^2 \ . \end{cases} \qquad (KGMP_f)$$

Assume A and φ depend on the sole spatial variables (static case), and look for standing waves solutions $u(x)e^{-i\omega t}$. The fourth equation gives that

$$\overline{\Delta}_g A + (q^2 u^2 + m_1^2) A = 0 \ .$$

This implies $A \equiv 0$ since $\int (\overline{\Delta}_g A, A) = \int |dA|^2$. Since $S = -\omega t$ the second equation is automatically satisfied. We get a reduced system.

The full system reduces to its first and third equation. Letting $\varphi = \omega v$, the reduced KGMP system consisting in the first and third equations is written as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{2^\star - 1} + \omega^2 \left(qv - 1\right)^2 u \\ \Delta_g v + \left(m_1^2 + q^2 u^2\right) v = qu^2 \ . \end{cases} \qquad (KGMP_r)$$

Here ω is the phase (or temporal frequency), $m_0, m_1 > 0$ are masses, $q > 0$ is an electric charge, $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator.

When we investigate $(KGMP_r)$ we talk about standing waves solutions for the full $(KGMP_f)$-system in static form.

By the $u^2 v$-term the second equation is subcritical when $n = 3$, critical when $n = 4$, and supercritical when $n \geq 5$.

I.4) The Einstein-Lichnerowicz equation :

Let (X, γ) be a Lorentzian manifold of dimension $n+1$. Typically $n=3$, but the dimension plays no specific role here. The Einstein equation is traditionally written in the form

$$G_{ij} = \frac{8\pi\mathcal{G}}{c^4} T_{ij} \ , \qquad \text{(EE)}$$

where G is the Einstein curvature tensor given by $G = Rc_\gamma - \frac{1}{2} S_\gamma \gamma$, T is the stress-energy tensor, $\mathcal{G}$ is the gravitational constant, and c is the speed of light. In $3+1$-dimensions we get the famous ten Einstein equations. In a scalar field theory, the stress energy tensor is given by a scalar field $\Psi : X \to \mathbb{R}$ and by

$$T_{ij} = \nabla_i \Psi \nabla_j \Psi - \frac{1}{2} |\nabla \Psi|_\gamma^2 \gamma_{ij} - V(\Psi)\gamma_{ij} \ ,$$

where V is a potential for Ψ. For the massive Klein-Gordon field theory we get that $V(\Psi) = \frac{1}{2} m^2 \Psi^2$, where m represents a mass. In case $\Psi \equiv 0$ and $V \equiv \Lambda$ we get the Einstein equations in vacuum space with Einstein's cosmological constant. We forget about the constant in front of the stress-energy tensor and fix $\frac{8\pi\mathcal{G}}{c^4} = 1$.

We let (M, g) be a closed Riemannian n-manifold, and we want to produce solutions of the Einstein equations on a Lorentzian manifold X having M as a spacelike hypersurface. The solution to this problem is given by the theory of Choquet-Bruhat and Geroch and is called the maximal Cauchy development of M.

We fix (M, g), and let also $K \in \otimes_s^{(2,0)}$ be a symmetric $(2, 0)$-tensor field in M, and $\psi, \pi : M \to \mathbb{R}$ be two smooth functions in M. We assume that g, K, ψ, and π satisfy the $1 + n$ following equations

$$\begin{cases} S_g - |K|_g^2 + (\mathrm{tr}_g K)^2 = \pi^2 + |\nabla\psi|_g^2 + 2V(\psi) \\ \nabla_g . K - \nabla \mathrm{tr}_g K = \pi \nabla\psi \ . \end{cases} \tag{CE}$$

Then the result of Choquet-Bruhat and Geroch essentially gives that we can solve the Einstein equations on a Lorentzian manifold (X, γ) such that M is a spacelike hypersurface in X, g is the induced metric on M by γ, K is the second fundamental form of M in X, and ψ and π are the scalar field data and its normalized time derivative. These equations are the constraint equations from general relativity.

The first equation is a scalar equation referred to as the Hamiltonian constraint. The second one is a vectorial equation referred to as the momentum constraint. In particular, we have $1+n$ equations and the unknowns are g, K, ψ and π. So $1+n$ equations, and

$$1+1+2\times\frac{n(n+1)}{2}=n^2+n+2$$

unknowns. The constraint equations are highly underdetermined. This is where the Lichnerowicz conformal method comes into the story, the objective being to reduce the number of unknowns.

The first idea is to fix the conformal class of the metric in M. This already kills $\frac{n(n+1)}{2}-1$ variables. We fix g_0 and search g under the form $g=u^{4/(n-2)}g_0$. Then, according to the equation which relates the scalar curvatures of two conformal metrics, the one we discussed before, the Hamiltonian constraint can be rewritten as

$$\begin{aligned} u^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\Delta_{g_0}u+S_{g_0}u\right) &= \pi^2+u^{-\frac{4}{n-2}}|\nabla\psi|^2_{g_0}+2V(\psi)\\ &\quad+u^{-\frac{8}{n-2}}|K|^2_{g_0}-\tau^2\ , \end{aligned}$$

where $\tau=\mathrm{tr}_g K$ is the trace of K with respect to g, which can be interpreted as the mean curvature of M in X.

We define P by the equation

$$K = u^{-2}P + \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 \ .$$

As we can check, P is trace-free with respect to g. We can compute $\nabla_{g_0}.K$ in terms of $\nabla_{g_0}.P$, and we get that the momentum constraints rewrite as

$$u^{-\frac{2n}{n-2}} \left(\nabla_{g_0}.P\right)_i = \frac{n-1}{n}(\nabla\tau)_i + \pi(\nabla\psi)_i$$

for all i. We define the conformal Killing operator $\mathcal{L}_{g_0}$ acting on vector fields X by

$$\left(\mathcal{L}_{g_0}X\right)_{ij} = \left(\nabla_j X\right)_i + \left(\nabla_i X\right)_j - \frac{2}{n}\left(\nabla_{g_0}.X\right) g_{ij}^0 \ ,$$

and we define the conformal Laplacian $\overrightarrow{\Delta}_{g_0}$ by

$$\overrightarrow{\Delta}_{g_0} X = \nabla_{g_0}.\left(\mathcal{L}_{g_0}X\right) \quad .$$

An equation like $\overrightarrow{\Delta}_{g_0} X = Y$ can be solved as soon as Y is orthogonal to the conformal Killing vector fields, a conformal Killing vector field being a solution of $\mathcal{L}_{g_0} Z \equiv 0$. On a compact manifold, Z is a conformal vector field iff $\overrightarrow{\Delta}_{g_0} Z \equiv 0$.

Since P is symmetric and trace-free, $\nabla_{g_0}.P$ is orthogonal to any conformal Killing vector field as we can check by writing that

$$\begin{aligned}
&\int_M (\nabla^\alpha P)_{i\alpha} Y^i dv_{g_0} \\
&= -\int_M P_{i\alpha} (\nabla^\alpha Y)^i dv_{g_0} \\
&= -\frac{1}{2}\int_M P_{i\alpha}\left((\nabla^\alpha Y)^i + (\nabla^i Y)^\alpha - \frac{2}{n}(\nabla_{g_0}.Y) g_0^{i\alpha}\right) dv_{g_0} \\
&= 0
\end{aligned}$$

as soon as $\mathcal{L}_{g_0} Y = 0$. Then we can write that

$$P = \sigma + \mathcal{L}_{g_0} X \ ,$$

where σ is a symmetric, trace-free and divergence-free $(2,0)$-tensor field, referred to as the TT-tensor. In particular, the momentum constraint equations rewrite as

$$\overrightarrow{\Delta}_{g_0} X = \left(\frac{n-1}{n}\nabla\tau + \pi\nabla\psi\right) u^{\frac{2n}{n-2}} \ .$$

Letting $\pi = u^{-\frac{2n}{n-2}}\tilde{\pi}$, the Hamiltonian and momentum constraint equations are written as

$$\begin{cases} \Delta_{g_0} u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \\ \overrightarrow{\Delta}_{g_0} X = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla\tau + \tilde{\pi}\nabla\psi \ , \end{cases} \tag{ELCE}$$

where

$$h = \frac{n-2}{4(n-1)} \left(S_{g_0} - |\nabla\psi|^2_{g_0} \right) \ ,$$

$$f = \frac{n-2}{4(n-1)} \left(2V(\psi) - \frac{n-1}{n}\tau^2 \right) \ ,$$

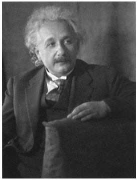
$$a = \frac{n-2}{4(n-1)} \left(|\sigma + \mathcal{L}_{g_0} X|^2_{g_0} + \tilde{\pi}^2 \right) \ .$$

These equations are the constraint equations of general relativity in Einstein-Lichnerowicz form. In the constant mean curvature case, referred to as the CMC case, when τ is taken to be constant, the two equations in (ELCE) are independent one of another.

The second equation in (ELCE) can be solved when $\tilde{\pi}\nabla\psi$ is orthogonal to the conformal Killing vector fields, or when there are no conformal Killing vector fields. The first equation, referred to as the Einstein-scalar field Lichnerowicz equation, reads as

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \,, \tag{EL}$$

The functions h, f, a are smooth. We always have $a \geq 0$ (and $a > 0$ when $\pi \neq 0$). In the CMC case, when $V \equiv C^t$ (cosmological constant), f is a constant (let's say 1 if τ is small in front of V).



Albert Einstein
1879-1955

André Lichnerowicz
1915-1998

I.5) The Kirchhoff equation :

The Kirchhoff equation goes back to Kirchhoff in 1883. It was proposed as an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. The equation is one dimensional, time dependent, and it was written as

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

where L is the length of the string, h is the area of cross-section, E is the young modulus (elastic modulus) of the material, ρ is the mass density, and P_0 is the initial tension. Almost one century later, Jacques Louis Lions returned to the equation and proposed an abstract framework for the general Kirchhoff equation in higher dimension with external force term. Lions equation was written as

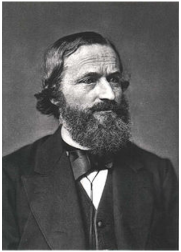
$$\frac{\partial^2 u}{\partial t^2} + \left(a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u)$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace-Beltrami (Euclidean) Laplacian.

Let (M, g) be a closed n-manifold, $n \geq 3$. Let $a, b > 0$ be positive real numbers. Let $h \in C^1(M, \mathbb{R})$ be a function. The Kirchhoff equation we consider is written as

$$\left(a + b \int_M |\nabla u|^2 dv_g \right) \Delta_g u + hu = u^{2^\star - 1} \ , \qquad (K)$$

where $2^\star$ is the critical Sobolev exponent. The equation is nonlocal in essence due to the multiplicative term in front of the Laplacian.



Gustav Kirchhoff
1824-1887

Thank you for your attention !