

Klein-Gordon-Maxwell-Proca systems
in the Riemannian setting

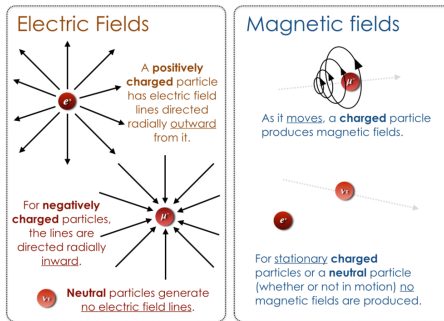
by
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From various works in collaboration with
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June 2018

I. The full KGMP-system.

The physics purpose of Klein-Gordon-Maxwell-Proca systems is that they provide a model for the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. In other words, the electromagnetic field is both generated by and drives the particle field.



Here : massive version of the more classical Klein-Gordon-Maxwell equations. The vector field representing the electromagnetic field has a mass.

Nonlinear Klein-Gordon total functional, minimum coupling rule

$$\partial_t \rightarrow \partial_t + iq\varphi \text{ and } \nabla \rightarrow \nabla - iqA,$$

where (φ, A) gauge potential representing the electromagnetic field, governed by the Maxwell-Proca Lagrangian. Consider the two Lagrangian densities

$$\mathcal{L}_{NKG}(\psi, \varphi, A)$$

$$= \frac{1}{2} \left| \left(\frac{\partial}{\partial t} + iq\varphi \right) \psi \right|^2 - \frac{1}{2} |(\nabla - iqA)\psi|^2 - \frac{m_0^2}{2} |\psi|^2 + \frac{1}{p} |\psi|^p$$

and

$$\mathcal{L}_{MP}(\varphi, A) = \frac{1}{2} \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - \frac{1}{2} |\nabla \times A|^2 + \frac{m_1^2}{2} |\varphi|^2 - \frac{m_1^2}{2} |A|^2 .$$

Here $\nabla \times = \star d$, $\star$ Hodge dual, d differentiation. Massive version of KGM theory. Here ψ matter field, m_0 its mass, q its charge, (A, φ) gauge potentials representing the electromagnetic vector field, m_1 is the Proca mass.

Consider the total action functional

$$\mathcal{S}(\psi, \varphi, A) = \int \int (\mathcal{L}_{NKG} + \mathcal{L}_{MP}) dv_g dt .$$

Write ψ in polar form as $\psi(x, t) = u(x, t)e^{iS(x, t)}$ for $u \geq 0$ and $u, S : M \times \mathbb{R} \rightarrow \mathbb{R}$. Then the total action rewrites as

$$\begin{aligned} \mathcal{S}(u, S, \varphi, A) = & \frac{1}{2} \int \int \left(\left(\frac{\partial u}{\partial t} \right)^2 - |\nabla u|^2 - m_0^2 u^2 \right) dv_g dt \\ & + \frac{1}{p} \int \int u^p dv_g dt \\ & + \frac{1}{2} \int \int \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u^2 dv_g dt \\ & + \frac{1}{2} \int \int \left(\left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times A|^2 + \frac{m_1^2}{2} |\varphi|^2 - \frac{m_1^2}{2} |A|^2 \right) dv_g dt . \end{aligned}$$

We can take the variation of $\mathcal{S}$ with respect to u , S , φ , and A . For instance, if we let ω_g be the volume form of (M, g) , then

$$\begin{aligned}
 \frac{1}{2} \left(\frac{d}{dA} \int |\nabla \times A|^2 \right) \cdot (B) &= \int (\star dA, \star dB) \omega_g && \text{(quadratic + } \nabla \times = \star d) \\
 &= (-1)^{n-1} \int (\star dA, (\star d \star) \star B) \omega_g && (\star \star = (-1)^{n-1} \text{ in } \Lambda^1) \\
 &= \int (\star dA, \delta \star B) \omega_g && (\delta = (-1)^{n-1} \star d \star \text{ in } \Lambda^{n-1}) \\
 &= \int (d \star dA, \star B) \omega_g && \text{(Stokes formula)} \\
 &= \int (\star \delta dA, \star B) \omega_g && (d \star = \star \delta \text{ in } \Lambda^2) \\
 &= \int (\star \delta dA) \wedge (\star \star B) && \text{(since } \alpha \wedge (\star \beta) = (\alpha, \beta) \omega_g \text{ in } \Lambda^p) \\
 &= (-1)^{n-1} \int (\star \delta dA) \wedge B && (\star \star = (-1)^{n-1} \text{ in } \Lambda^1) \\
 &= \int (\delta dA, B) \omega_g && (\alpha \wedge \beta = (-1)^{n-1} \beta \wedge \alpha \text{ for } \alpha \in \Lambda^{n-1}, \beta \in \Lambda^1)
 \end{aligned}$$

In particular,

$$\frac{1}{2} \left(\frac{d}{dA} \int |\nabla \times A|^2 \right) .(B) = \int (\overline{\Delta}_g A, B)$$

for all B , where $\overline{\Delta}_g = \delta d$, δ codifferential. Taking the variation of

$$\mathcal{S}(\psi, \varphi, A) = \int \int (\mathcal{L}_{NKG} + \mathcal{L}_{MP}) dv_g dt .$$

with respect to u , S , φ , and A , we then get four equations which are written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) + m_1^2 A = q (\nabla S - qA) u^2 . \end{cases} \quad (KGMP)$$

This is the nonlinear Klein-Gordon-Maxwell-Proca system. As $m_1 \rightarrow 0$ (or letting $m_1 = 0$), the nonlinear KGMP system reduces to the nonlinear Klein-Gordon-Maxwell system.

This kind of system, without the Proca contribution (take $m_1 = 0$), without external force (cancel the u^{p-1} -term), but with the full time dependency, has been investigated by several people (the list is certainly far from being exhaustive) like Cuccagna, Linblad, Keel, Klainerman, Machedon, Rodnianski, Roy, Selberg, Sterbenz, Tataru, Tesfahun and Tao (and they prove local well-posedness, global well-posedness $n = 3$, global well-posedness for small initial data $n \geq 4$, global well-posedness below the energy norm, critical dimension $n = 4$, asymptotic behavior).

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 A = q (\nabla S - qA) u^2 . \end{cases} \quad (KGMP)$$

II. Why do we refer to Maxwell-Proca ?

Somehow we go backward with respect to what dispersive mathematicians do. Assume $n = 3$. Let the electric field E , the magnetic induction H , the charge density ρ , and the current density J be given by

$$E = - \left(\frac{\partial A}{\partial t} + \nabla \varphi \right) ,$$

$$H = \nabla \times A ,$$

$$\rho = - \left(\frac{\partial S}{\partial t} + q\varphi \right) qu^2 ,$$

$$J = (\nabla S - qA) qu^2 .$$

The two last equations in (*KGMP*) give rise to the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J .

We get for free the second pair of the Maxwell-Proca equations.

In other words the two last equations in the $(KGMP)$ -system can be rewritten as

$$\begin{aligned}\nabla.E &= \rho - m_1^2\varphi , \\ \nabla \times H - \frac{\partial E}{\partial t} &= J - m_1^2 A , \\ \nabla \times E + \frac{\partial H}{\partial t} &= 0 , \quad \nabla.H = 0 .\end{aligned}$$

The first equation in the $(KGMP)$ -system is the nonlinear Klein-Gordon matter equation. Namely

$$\frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \frac{\rho^2 - |J|^2}{q^2 u^3} .$$

The second equation in the $(KGMP)$ -system is the charge continuity equation $\frac{\partial \rho}{\partial t} + \nabla.J = 0$, which is equivalent to the Lorentz condition

$$\nabla.A + \frac{\partial \varphi}{\partial t} = 0 .$$

The $(KGMP)$ -system is equivalent to this system of 6 equations.

The equivalence between the charge continuity equation and the Lorentz condition involves only basic computations and uses in a fundamental way the condition $m_1 \neq 0$. The Maxwell-Proca equations are written as

$$\begin{aligned}\nabla \cdot E &= \rho - m_1^2 \varphi, \quad \nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A, \\ \nabla \times E + \frac{\partial H}{\partial t} &= 0, \quad \nabla \cdot H = 0.\end{aligned}$$

The charge continuity equation states that $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$. Taking the derivation of the first Maxwell equation with respect to time, and the divergence of the second equation,

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot J &= \nabla \cdot \frac{\partial E}{\partial t} + m_1^2 \frac{\partial \varphi}{\partial t} + \nabla \cdot (\nabla \times H) - \nabla \cdot \frac{\partial E}{\partial t} + m_1^2 \nabla \cdot A \\ &= m_1^2 \left(\nabla \cdot A + \frac{\partial \varphi}{\partial t} \right)\end{aligned}$$

since $\nabla \cdot (\nabla \times H) = \delta(\star d)H$, $\delta = \star^{-1}d\star$ in Λ^1 , $\star\star = 1$ in Λ^2 , and $d^2 = 0$ so that $\nabla \cdot (\nabla \times H) = 0$. The condition $m_1 \neq 0$ breaks the gauge invariance and enforces the Lorentz gauge.

III. The physics behind the Proca contribution

The Maxwell equations in Proca form are

$$\begin{aligned}\nabla \cdot E &= \rho - m_1^2 \varphi, \quad \nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A, \\ \nabla \times E + \frac{\partial H}{\partial t} &= 0, \quad \nabla \cdot H = 0.\end{aligned}$$

They reduce to the Maxwell equations as $m_1 \rightarrow 0$. Proca (1936) was using the Lorentz formalism. Under this form, referred to as the “modern format”, the equations appeared for the first time in a paper by Schrödinger : “*The earth's and the sun's permanent magnetic fields in the unitary field theory*” (1943). These equations have been discussed by several physicists including, in addition to Proca and Schrödinger, people like De Broglie, Pauli, Yukawa, and Stueckleberg... The whole point in these theories is that m_1 is nothing but than the mass of the photon : we are talking about a theory where photons have a mass (“the photon is the quantum of the electromagnetic field including electromagnetic radiation such as light, and the force carrier for the electromagnetic force”).

VIII.

THE EARTH'S AND THE SUN'S PERMANENT MAGNETIC FIELDS IN THE UNITARY FIELD THEORY.

(From the Dublin Institute for Advanced Studies.)

By ERWIN SCHRÖDINGER.

[Read 28 JUNE. Published 29 NOVEMBER, 1943.]

§ 1. SURVEY.

For not excessively strong electromagnetic fields in empty space and neglecting gravitation the Unitary Field Theory¹ gives the equations ($c = 1$)

$$\begin{aligned} H &= \text{curl } A \\ E &= -\dot{A} - \text{grad } V \\ \text{curl } H - \dot{E} &= -\mu^2 A \\ \text{div } E &= -\mu^2 V \end{aligned} \tag{1}$$

and suggests that the constant μ^{-1} be not cosmically large (in which case the equations boil down to Maxwell's) but very roughly speaking of the order of the radius of the earth.



Erwin Schrödinger
1887 – 1961

The Earth's and the Sun's Permanent Magnetic
Fields in the Unitary Field Theory

Erwin Schrödinger

*Proceedings of the Royal Irish Academy. Section
A: Mathematical and Physical Sciences*

Vol. 49 (1943/1944), pp. 135-148

Louis de Broglie, 1950

Sur une forme nouvelle de la théorie du champ soustractif

A partir de 1934, l'auteur du présent article a développé une forme nouvelle de la théorie quantique du champ électromagnétique qu'il a appelé "la Mécanique ondulatoire du photon" et qui présentait à ses yeux l'avantage de faire plus clairement rentrer la théorie quantique des champs dans le cadre général de la Mécanique ondulatoire des particules à spin. Dans cette théorie, qui a été exposée dans plusieurs Ouvrages, il a été attribué au photon une masse propre extrêmement petite, mais non nulle, et nous avons été ainsi conduit dès 1934 à prendre comme équations de la particule de spin 1 des équations qui, mises sous forme vectorielle, sont des équations du type classique de Maxwell complétées par de petits termes contenant la masse propre. Des équations de même forme ont été ensuite proposées, en 1936, par M. Alexandre Proca, et on leur donne aujourd'hui dans la théorie du méson le nom d'équations de Proca. En somme ces équations sont les équations générales des particules de spin 1.

LE JOURNAL DE PHYSIQUE

ET

LE RADIUM

SUR UNE FORME NOUVELLE DE LA THÉORIE DU « CHAMP SOUSTRACTIF »

Par M. LOUIS DE BROGLIE.

Sommaire. — Après avoir fait un nouvel exposé des principes de la « théorie du champ soustractif » qu'il a récemment proposée, l'auteur montre comment on peut dans cette théorie exprimer la liaison entre les corpuscules élémentaires et les champs photoniques ou mésoniques, avec lesquels ils sont en interaction et précise certains aspects intéressants de cette interaction. Il rappelle ensuite comment on peut donner aux éléments des matrices d'interaction une forme qui soit satisfaisante du point de vue relativiste et étudie l'influence des corrections introduites ainsi sur la convergence des intégrales qui se présentent dans le calcul des énergies propres. Utilisant ensuite des calculs effectués par M. R. P. Feynman, il détermine d'une façon approximative la valeur des incréments de masse qui résultent pour l'électron de son interaction avec le champ photonique et un champ mésonique : les résultats obtenus paraissent acceptables.

1. Introduction. — A partir de 1934, l'auteur du présent article a développé une forme nouvelle de la théorie quantique du champ électromagnétique qu'il a appelée « la Mécanique ondulatoire du photon » et qui présentait à ses yeux l'avantage de faire plus clairement rentrer la théorie quantique des champs dans le cadre général de la Mécanique ondulatoire des particules à spin. Dans cette théorie qui a été exposée dans plusieurs Ouvrages [1, 2, 3, 4], il a été attribué au photon une masse propre extrêmement petite, mais non nulle, et nous avons été ainsi conduit dès 1934 [1] à prendre comme équations

électrisées, résultat inadmissible qui embarrassait les théoriciens depuis plus de 20 ans.

Récemment, nous avons repris sous une forme nouvelle une tentative (théorie du champ soustractif) faite naguère par M. Stueckelberg pour lever cette difficulté [5] et reprise ensuite de diverses façons par MM. Bopp [6], Pais [7] et Feynman [8]. Nous avons exposé nos idées nouvelles dans quatre Notes aux *Comptes rendus de l'Académie des Sciences* [9] et dans un *Mémoire de Portugaliae mathematica* [10].

Sans reprendre ici le détail des calculs donnés

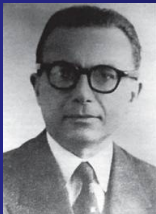
What about nowadays ? Well, people are still pretty much enthusiastic. . .

Eric Adelberger, Gia Dvali, Andrei Gruzinov
Photon-mass bound destroyed by vortices
Phys. Rev. Lett., 2007.

The possibility of a nonzero photon mass remains one of the most important issues in physics , as it would shed light on fundamental questions such as charge conservation, charge quantization, the possibility of charged black holes and magnetic monopoles, etc. The most stringent upper bounds on the photon mass listed by the particle Data Group, $m < 3 \times 10^{-27}$ eV and $m < 2 \times 10^{-16}$ eV, are based on the assumption that a massive photon cause large-scale magnetic fields to be accompanied by an energy density

$$m_A^2 \tilde{A}_\mu \tilde{A}^\mu$$

associated with the Proca field $\tilde{A}_\mu$ that describes the massive photon.



Alexandru Proca
1897-1955



Louis de Broglie
1892-1987

Some possible references :

- [1] G.T.Gillies, J.Luo, L.C.Tu, The mass of the photon, Report on Progress in Physics, 68, 2005, 77–130.
- [2] A.S.Goldhaber, M.M.Nieto, Photon and Graviton mass limits, Reviews of Modern Physics, 82, 2010, 939–979.
- [3] H.Ruegg M. Ruiz-Altaba, The Stueckleberg field, International Journal of Modern Physics A, 19, 2004, 3265–3348.

IV. A drastic elliptic reduction of the equations

Return to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 A = q (\nabla S - qA) u^2 . \end{cases} \quad (KGMP)$$

Assume A and φ depend on the sole spatial variables (static case), and look for standing waves solutions $u(x)e^{-i\omega t}$. The fourth equation gives that

$$\overline{\Delta}_g A + (q^2 u^2 + m_1^2) A = 0 .$$

This implies $A \equiv 0$ since $\int (\overline{\Delta}_g A, A) = \int |dA|^2$. Since $S = -\omega t$ the second equation is automatically satisfied. The full system reduces to its first and third equation. Letting $\varphi = \omega v$, we are led to the system

$$\begin{cases} \Delta_g u + \Phi(x, v)u = u^{p-1} \\ \Delta_g v + (m_1^2 + q^2 u^2) v = q u^2, \end{cases} \quad (S_\omega)$$

where $\Phi(x, v) = m_0^2 - \omega^2(qv - 1)^2$. Let $2^* = \frac{2n}{n-2}$ be the critical Sobolev exponent for H^1 . With respect to the first equation in (S_ω) the system is energy subcritical when $p < 2^*$ and energy critical when $p = 2^*$. The second equation is subcritical when $n = 3$, critical when $n = 4$, and supercritical when $n \geq 5$.

The system is truly coupled thanks to the Proca mass m_1 . If $m_1 = 0$ then $v = 1/q$ and system $\rightarrow \Delta_g u + m_0^2 u = u^{p-1}$.

In the case of $\mathbb{R}^3$, when $p < 2^*$ and $m_1 = 0$ (here again the list is certainly far from being exhaustive) : D'Aprile, d'Avenia, Azzollini, Benci, Bonanno, Cassani, Fortunato, Georgiev, Ghimenti, Mugnai, Pisani, Pomponio, Siciliano, Vaira, Visciglia (existence of a solution). The critical case in the closed setting has been investigated by Druet, H., Thizy, Truong, Vétois, Wei (2010-2015, compactness, non compactness, phase compensation, existence of a solution).

Examples of results in the $\mathbb{S}^3$ and $\mathbb{S}^4$ model cases :

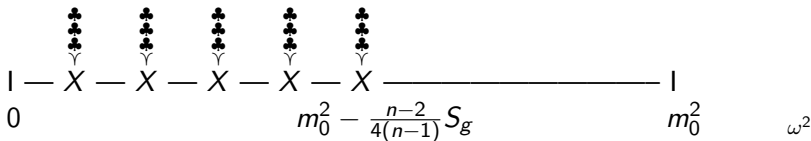
D = Druet, T = Truong, V = Vétois, W = Wei ($0 \leq \omega^2 \leq m_0^2$, S_g scalar curvature)

[1] Case of $\mathbb{S}^3$: ($\frac{n-2}{4(n-1)} S_g = \frac{3}{4}$)

Resonant States^{DHW}

A priori bounds^{DH}

(full compactness)



[2] Case of $\mathbb{S}^4$: ($\frac{n-2}{4(n-1)} S_g = 2$)

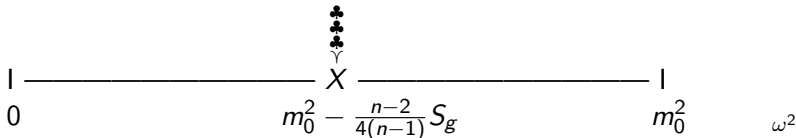
Phase stability^{DHV}

A priori bounds^{HT}

(compactness of finite energy standing waves)

(full compactness)

Resonant states^{DHV}



V. A less drastic reduction of the equations

Return again to the original system

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left((\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\left(\frac{\partial A}{\partial t} + \nabla \varphi \right) \right) + m_1^2 A = q (\nabla S - qA) u^2 . \end{cases} \quad (KGMP)$$

Still assume that A and φ depend on the sole spatial variables (static case), but now look for solutions like

$$\Psi(x, t) = u(x) e^{i(S(x) - \omega t)} .$$

These solutions for such kind of systems were introduced in Benci and Fortunato [CMP, 2010] with the specific choice of S given by

$$S(x) = \ell \operatorname{Im} \ln(x_1 + ix_2) ,$$

where ℓ is an integer (spinning Q -ball). In the specific case of the KGMP-system : D'Avenia, Mederski and Pomponio [J. Math. Phys., 2017].

We slightly differ from this approach here by considering that S is a free parameter in the system. When we plugg Ψ in the system, and we assume that A and φ do not depend on t , we do get the following static system of equations

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \left((q\varphi - \omega)^2 - |\nabla S - qA|^2 \right) u \\ \nabla \cdot ((\nabla S - qA) u^2) = 0 \\ \Delta_g \varphi + m_1^2 \varphi + q(q\varphi - \omega) u^2 = 0 \\ \overline{\Delta}_g A + m_1^2 A = q(\nabla S - qA) u^2 . \end{cases} \quad (KGMP_s)$$

In the limit case $m_1 = 0$, the second equation is automatically satisfied (since $\delta \overline{\Delta}_g = 0$). When $m_1 \neq 0$, thanks to the fourth equation in this system, still since $\delta \overline{\Delta}_g = 0$, the second equation in the system can be omitted and replaced by the Coulomb gauge equation $\delta A = 0$. One more remark is that the Coulomb gauge equation $\delta A = 0$ implies that $\overline{\Delta}_g A = \Delta_g A$, where $\Delta_g = d\delta + \delta d$ is the usual Hodge-de Rham Laplacian on forms.

VI. Let's discuss at least one theorem ...

In what follows $n = 3$ and the system we investigate is written as

$$\begin{cases} \Delta_g u + \Phi(x, v, A)u = u^{p-1} \\ \Delta_g v + (b + q^2 u^2) v = qu^2 \\ \Delta_g A + bA = q(\nabla S - qA)u^2, \end{cases} \quad (E)$$

where $q > 0$,

$$\Phi(x, v, A) = a - \omega^2 (qv - 1)^2 + |\nabla S - qA|^2,$$

$\omega \in \mathbb{R}$, $a, b, S \in C^\infty(M)$ are smooth functions with $a, b > 0$ in M , $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator when acting on functions u and v in the two first equations, $\Delta_g = \delta d + d\delta$ is the Hodge-de Rham Laplacian when acting on 1-forms A in the third equation and the pure power nonlinearity $p \in (2, 6]$.

Here a, b, S and ω are parameters. The unknowns are u, v and A . We have let $\varphi = \omega v$. Also b is now a function which extends the role of m_1^2 , and a is a function which extends the role of m_0^2 .

Theorem (H.-Thizy, 2017)

(M, g) smooth closed 3-manifold, $q > 0$ real, $a, b, S \in C^\infty$ with $a > 0$ and $b \geq 0$ in M . Assume that $\text{Ric}_g + bg > 0$ in M . Let $\omega \in (-\min_M \sqrt{a}, +\min_M \sqrt{a})$ and $p \in (2, 6]$. Assume that when $p = 6$ is critical,

$$a - \omega^2 + |\nabla S|^2 < \Lambda_g$$

in M if $b \not\equiv 0$, or that $a + |\nabla S|^2 < \Lambda_g$ in M if $b \equiv 0$, where $\Lambda_g \in C^\infty$, $\Lambda_g > 0$, is such that $\Delta_g + \Lambda_g$ has positive mass. Then

$$\begin{cases} \Delta_g u + \Phi(x, v, A)u = u^{p-1} \\ \Delta_g v + (b + q^2 u^2) v = qu^2 \\ \Delta_g A + bA = q(\nabla S - qA)u^2, \end{cases} \quad (E)$$

possesses a smooth nontrivial solution (u, v, A) with $u > 0$ and $v > 0$, and the set $\mathcal{S}$ of solutions (u, v, A) of (E) with $u \geq 0$ and $v \geq 0$ is compact in the C^2 -topology.

The existence part in the theorem misses the gauge equation $\delta A = 0$. On the other hand the compactness part in the theorem obviously remains valid for the full system $(E) + \delta A = 0$.

Corollary (Almost the gauge equation)

Let $\theta \in (0, 1)$. Let a, ω satisfy that $a - \omega^2 < \Lambda_g$ in M if $b \not\equiv 0$, or that $a < \Lambda_g$ if $b \equiv 0$, where $\Lambda_g > 0$ is a smooth positive function such that $\Delta_g + \Lambda_g$ has positive mass. There exists $\varepsilon_0 > 0$ such that if

$$\|\nabla S\|_{C_R^{0,\theta}} < \varepsilon_0 ,$$

then (E) possesses a smooth nontrivial solution (u_S, v_S, A_S) and there holds that $\delta A_S \rightarrow 0$ in C_R^1 as $S \rightarrow 1$ in $C_R^{1,\theta}$.

VII. Few words on the proof of the theorem

There are several levels in the proof of the theorem. Let H_R^1 be the Sobolev space of functions in L^2 with one derivative in L^2 and H_V^1 be the corresponding space for 1-forms.

Level 1 : functional analysis.1

Assume that $\text{Ric}_g + bg > 0$ in M , in the sense of bilinear forms, where Ric_g is the Ricci curvature of g . For any $u \in H_R^1$, there exists a unique $A_u \in H_V^1$ such that

$$\Delta_g A_u + (b + q^2 u^2) A_u = q(\nabla S)u^2 .$$

There exists a constant $C > 0$, depending only on q and S , such that $|A_u| \leq C$ in M for all $u \in H_R^1$. The map $A : H_R^1 \rightarrow H_V^1$ is differentiable as well as the map $\Phi : H_R^1 \rightarrow \mathbb{R}$ defined by $\Phi(u) = \int_M (\nabla S - qA_u, \nabla S) u^2 dv_g$. The differential of Φ is given for all $u, h \in H_R^1$ by

$$D\Phi(u).(h) = 2 \int_M |\nabla S - qA_u|^2 u h dv_g .$$

Level 2 : functional analysis.2

Assume that $b > 0$ in M . For any $u \in H_R^1$, there exists a unique $v_u \in H_R^1$ such that

$$\Delta_g v_u + (b + q^2 u^2) v_u = q u^2 .$$

There holds that $0 \leq v_u \leq \frac{1}{q}$ for all $u \in H_R^1$. The map $v : H_R^1 \rightarrow H_R^1$ is differentiable as well as the map $\Psi : H_R^1 \rightarrow \mathbb{R}$ defined by $\Psi(u) = \int_M (1 - q v_u) u^2 dv_g$. The differential of Ψ is given by

$$D\Psi(u).(h) = 2 \int_M (1 - q v_u)^2 u h dv_g$$

for all $u, h \in H_R^1$.

$\Rightarrow$ if (u, v, A) solves (E) , then $v = v_u$ and $A = A_u$.

Formally, solutions of our system are critical points of the functional $I : H_R^1 \times H_R^1 \times H_V^1 \rightarrow \mathbb{R}$ given by

$$\begin{aligned} I(u, v, A) = & \frac{1}{2} \int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2} \int_M |\nabla v|^2 dv_g + \frac{1}{2} \int_M |dA|^2 dv_g \\ & + \frac{1}{2} \int_M |\delta A|^2 dv_g + \frac{1}{2} \int_M a u^2 dv_g - \frac{\omega^2}{2} \int_M b v^2 dv_g \\ & - \frac{\omega^2}{2} \int_M u^2 (1 - qv)^2 dv_g + \frac{1}{2} \int_M b |A|^2 dv_g \\ & + \frac{1}{2} \int_M |\nabla S - qA|^2 u^2 dv_g - \frac{1}{p} \int_M |u|^p dv_g \end{aligned}$$

This is a highly competitive functional !

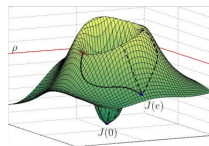
Level 3 : variational analysis.

We define the functional $I_p : H_R^1 \rightarrow \mathbb{R}$ by

$$I_p(u) = \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{1}{2} \int_M a u^2 dv_g - \frac{\omega^2}{2} \int_M (1 - q v_u) u^2 dv_g \\ + \frac{1}{2} \int_M (\nabla S - q A_u, \nabla S) u^2 dv_g - \frac{1}{p} \int_M (u^+)^p dv_g ,$$

where A_u and v_u are as before, and where $u^+ = \max(0, u)$. The functional I_p is differentiable and if u is a critical point of I_p , then (u, v_u, A_u) is a smooth solution of (E) with $u, v \geq 0$.

$\Rightarrow$ existence of a solution in the subcritical case by using MPL (by compactness of Sobolev embeddings).



Level 4 : existence in the critical case.

MPL + Schoen's test function analysis. Given $u_0 \in H_R^1$, define

$$c(u_0) = \inf_{P \in \mathcal{P}} \max_{u \in P} I_6(u) ,$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to u_0 and I_6 is the functional I_p in the critical case $p = 6$. Picking u_0 as a Schoen's test function, assuming that we do have the assumption on a , ω and S in the theorem, there exists $\delta_0 > 0$ such that

$$\delta_0 \leq c(u_0) \leq \frac{1}{3K_3^3} - \delta_0 ,$$

where K_3 is the sharp constant in the Euclidean Sobolev inequality. Then the Aubin-Brézis-Nirenberg arguments work for MPL. Below the best constant $\Rightarrow$ compactness $\Rightarrow$ existence of a nontrivial solution (but now) in the critical case.

Level 5 : Compactness in the subcritical case.

Let $(u_\alpha, v_\alpha, A_\alpha)_\alpha$ be a sequence of solutions of our equations. Then $v_\alpha = v_{u_\alpha}$ and $A_\alpha = A_{u_\alpha}$. The whole point then is to control u_α . Assume by contradiction that

$$\max_M u_\alpha \rightarrow +\infty$$

as $\alpha \rightarrow +\infty$. Rescale the u_α 's properly at their maximum point, go the limit and get a nontrivial nonnegative solution in $\mathbb{R}^3$ of

$$\Delta u = u^{p-1} .$$

By Gidas-Spruck such an equation does not have any nonnegative nontrivial solution. The u_α 's are bounded in L^∞ . We conclude to a C^2 -convergence with the Gilbarg-Trüdinger standard elliptic PDEs estimates. Q.E.D.

The 3-dimensional strategy we discuss here goes back to Li-Zhu and Schoen's work on the 3-dimensional Yamabe problem.

Level 6 : Compactness in the critical case.1

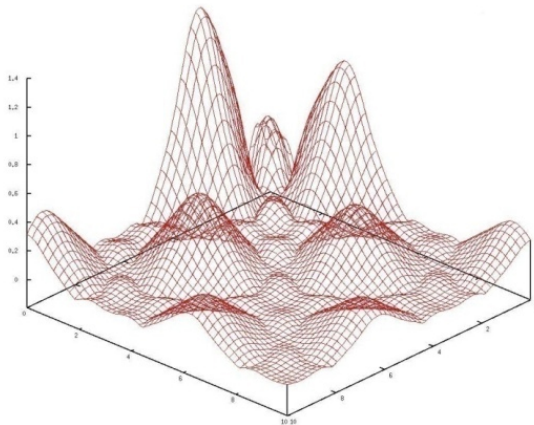
Prove that there are no accumulations of bubbles.

The assumptions are those of the theorem. Let $(u_\alpha, v_\alpha, A_\alpha)_\alpha$ be a sequence of solutions of our equations. Then $v_\alpha = v_{u_\alpha}$ and $A_\alpha = A_{u_\alpha}$. The whole point then is to control u_α . Let $(x_\alpha)_\alpha$ be a sequence of model points for the blow-up. Basic scaling :

$$\mu_\alpha^{\frac{1}{2}} u_\alpha (\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$$

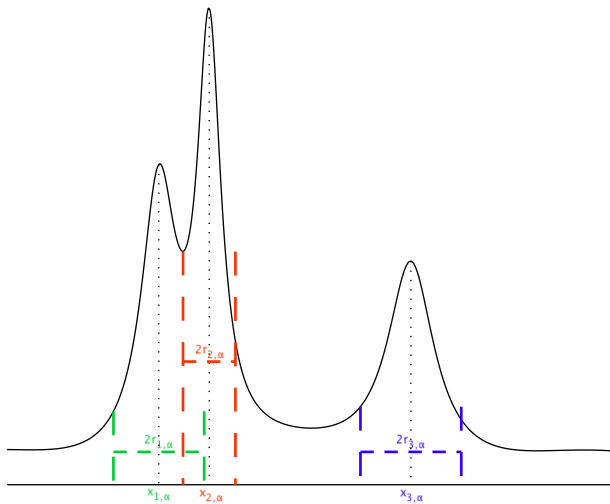
in $C_{loc}^1(\mathbb{R}^3)$ as $\alpha \rightarrow +\infty$, where $\mu_\alpha = u_\alpha(x_\alpha)^{-2}$ ($\mu_\alpha \rightarrow 0$). This is nothing but the Gidas-Spruck seminal scaling argument when transposed to the critical case.

We could have a configuration like



A lot of bubbles (possibly infinite number) with cluster's type configurations (groups of bubbles interacting one with another).

We modelize the configuration by



Make a list of possible (theoretical) blow-up points $(x_{i,\alpha})_\alpha$. Let d_α be the minimum distance between two blow-up points. Assume (free) that $d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha})$ is the distance between the two first points. Assume that $d_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$.

Rescale the u_α 's at $x_{1,\alpha}$, with a scale like the $r_{1,\alpha}$ that we do have in the picture. The $r_{1,\alpha}$'s turn out to be of the order of the d_α 's, and there holds that

$$d_\alpha \mu_{1,\alpha}^{-\frac{1}{2}} u_\alpha \left(\exp_{x_{1,\alpha}}(d_\alpha x) \right) \rightarrow \frac{\sqrt{3}}{|x|} + \mathcal{H}(x)$$

in $C_{loc}^2(B_0(2) \setminus \{0\})$ as $\alpha \rightarrow +\infty$, where $\mathcal{H}$ is a harmonic function in $B_0(2)$ which satisfies that $\mathcal{H}(0) \leq 0$.

There are blow-up points which are at a distance d_α of $x_{1,\alpha}$ and others which are not. Rescale the u_α 's, v_α 's and A_α 's at $x_{1,\alpha}$ with an order like d_α . The v_α 's and A_α 's are bounded (Levels 1 & 2) $\Rightarrow$ they do not count. Rescaling the u_α 's, using the formula in the preceding slide, and going to the limit in the equations we do get a convergence of the rescaled u_α 's to a function like

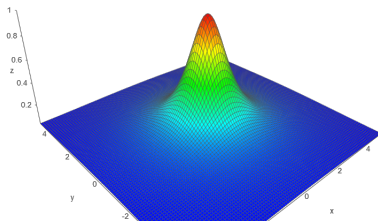
$$\tilde{G} = \sum_{i=1}^N \frac{\Lambda_i}{|x - \tilde{x}_i|} + \hat{\mathcal{H}}(x)$$

where $\hat{\mathcal{H}}$ is harmonic, the Λ_i 's are positive and the $\tilde{x}_i$'s are rescaling of the blow-up points which interact with $x_{1,\alpha}$. By assumption the $x_{1,\alpha}$'s and $x_{2,\alpha}$'s should be counted in this story, so at least $N \geq 2$. And also $\tilde{x}_1 = 0$. By the preceding slide

$$\sum_{i=2}^N \frac{\Lambda_i}{|\tilde{x}_i|} + \hat{\mathcal{H}}(0) \leq 0 .$$

This is impossible since $\tilde{G} \geq 0$ implies that $\hat{\mathcal{H}}(0) \geq 0$. As a consequence $d_\alpha \not\rightarrow 0$. This is step 1 in the analysis.

In other words we just need to handle the case of isolated bubbles like



There is necessarily a finite number of such objects (possibly many, all isolated).

Level 7 : Compactness in the critical case.2
Conclude in the case of isolated bubbles.

By the preceding step in the blow-up analysis, we do have bounded energy, we recover a well defined H^1 -Struwe type decomposition for the u_α 's $\Rightarrow$ we recover well defined blow-up points for the u_α 's, these blow-up points are isolated. Plugg our u_α 's into a Riemannian version of the Pohozaev identity. Still assuming that $\max_M u_\alpha \rightarrow +\infty$ we rescale properly the equation satisfied by the u_α 's at the blow-up point with the maximum weight (lowest bubble). The v_α 's and A_α 's are smoothed by their equations (they converge). We get a convergence for the rescaled u_α 's to a sum of Green's functions

$$\hat{G} = 4\pi\sqrt{3} \sum \mu_i^{1/2} G(x_i, \cdot) ,$$

where $\mu_i > 0$ and G is the Green function of $\Delta_g + \Phi_\infty$, where

$$\Phi_\infty = a - \omega^2 + |\nabla S|^2 .$$

The Green's function can be written as

$$G(x, y) = \frac{1}{\omega_2 d_g(x, y)} + R(x, y) ,$$

where R is continuous on the whole of $M \times M$. The Pohozaev identity implies that

$$\mu_i^{1/2} \sum_{j=1}^N \mu_j^{1/2} R(x_j, x_i) \leq 0 .$$

By construction $R(x_j, x_i) \geq 0$ for $i \neq j$. By our assumption that $\Phi_\infty \leq \Lambda_g$, where Λ_g is such that $\Delta_g + \Lambda_g$ has positive mass, we do get that $R(x_i, x_i) > 0$. A contradiction.

Thus $\max_M u_\alpha \not\rightarrow +\infty$ and the u_α 's are bounded in L^∞ . We conclude to a C^2 -convergence with the Gilbarg-Trüdinger standard elliptic PDEs arguments. Q.E.D.

Thank you for your attention