Stationary Kirchhoff systems in closed manifolds

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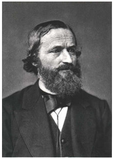
Geometric non-linear analysis :
Conference on the occasion of
Michael Struwe's 60th birthday

ETH Zürich
June 2015

I. The Kirchhoff equations



The Kirchhoff equation goes back to Kirchhoff in 1883.
It was proposed as an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. The equation is one dimensional, time dependent, and it was written as

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

where L is the length of the string, h is the area of cross-section, E is the young modulus (elastic modulus) of the material, ρ is the mass density, and P_0 is the initial tension. Almost one century later, Jacques Louis Lions returned to the equation and proposed an abstract framework for the general Kirchhoff equation in higher dimension with external force term. Lions equation was written as

$$\frac{\partial^2 u}{\partial t^2} + \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u)$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace-Beltrami (Euclidean) Laplacian.

Let (M^n, g) be a closed Riemannian manifold, $n \geq 3$. Let $p \geq 1$ integer, $a, b > 0$ positive real numbers, $A : M \to M_s^p(\mathbb{R})$ be a C^1-map from M to the set of symmetric $p \times p$ matrices with real entries. Consider the following Kirchhoff system of p equations :

$$\left(a + b\sum_{j=1}^{p} \int_M |\nabla u_j|^2 dv_g\right)\Delta_g u_i + \sum_{j=1}^{p} A_{ij} u_j = |U|^{2^\star - 2} u_i \qquad (S)$$

for all $i = 1, \ldots, p$, $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator, the A_{ij}'s are the components of A, $U = (u_1, \ldots, u_p)$,

$$|U| = \sqrt{\sum_{j=1}^{p} u_j^2}\ ,$$

and we require that $u_i \geq 0$ in M for all i. The system is strongly coupled. Here $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent.

Example 1 : When $p = 1$, we are discussing the single equation

$$\left(a + b \int_M |\nabla u|^2 dv_g\right) \Delta_g u + Au = u^{2^\star - 1} ,$$

where $A : M \to \mathbb{R}$ is a function.

Example 2 : When $p = 2$, we are discussing the system :

$$\begin{cases} \left(a + b \sum_{i=1}^2 \int_M |\nabla u_i|^2 dv_g\right) \Delta_g u_1 + A_{11} u_1 + A_{12} u_2 = |U|^{2^\star - 2} u_1 \\ \left(a + b \sum_{i=1}^2 \int_M |\nabla u_i|^2 dv_g\right) \Delta_g u_2 + A_{21} u_1 + A_{22} u_2 = |U|^{2^\star - 2} u_2 , \end{cases}$$

where $U = (u_1, u_2)$, the norm $|U|^2 = u_1^2 + u_2^2$, and the map $A : M \to M_s^2(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with $A_{12} = A_{21}$. Same story for $p = 3, 4, \ldots$.

Rk : (S) is nonlocal by $K(U) = a + b \sum_{i=1}^p \int_M |\nabla u_i|^2 dv_g$ which stands in front of the Laplacians in the equations.

II. The critical Euclidean p-map equation

The vector valued Euclidean critical equation in $\mathbb{R}^n$ we consider is

$$\Delta u_i = |U|^{2^\star - 2} u_i \tag{E}$$

for all $i = 1, \ldots, p$, where $\Delta = -\sum_{i=1}^n \partial_i^2$, and $|U|^2 = \sum u_i^2$.

Theorem A : (Druet-H.-Vétois, 2010).

Let $p \geq 1$ and U be a nonnegative nontrivial C^2-solution of (E). Then there exist $a \in \mathbb{R}^n$, $\mu > 0$, and $\Lambda \in S_+^{p-1}$, such that

$$U(x) = \left(\frac{\mu}{\mu^2 + \frac{|x-a|^2}{n(n-2)}} \right)^{(n-2)/2} \Lambda$$

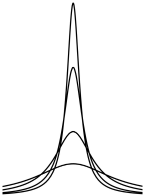
for all $x \in \mathbb{R}^n$, where S_+^{p-1} consists of the elements $(\Lambda_1, \ldots, \Lambda_p)$ in S^{p-1}, the unit sphere in $\mathbb{R}^p$, which are such that $\Lambda_i \geq 0$ for all i.

When $p = 1$ this is the Caffarelli-Gidas-Spruck theorem. (E) has precisely a "$1 + n + (p-1)$"-family of nonnegative solutions.

Definition of a bubble : *A bubble in the vector valued setting is a sequence* $(\mathcal{B}_\alpha)_\alpha$ *of p-maps,* $\mathcal{B}_\alpha : M \to \mathbb{R}^p$*, which are given by*

$$\mathcal{B}_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha,x)^2}{n(n-2)}} \right)^{\frac{n-2}{2}} \Lambda$$

for all $x \in M$ *and all* α*, where* $(x_\alpha)_\alpha$ *is a converging sequence of points in* M*,* $(\mu_\alpha)_\alpha$ *is a sequence of positive real numbers converging to* 0*, and* $\Lambda \in S_+^{p-1}$*.*



$(|\mathcal{B}_\alpha|$ as $\alpha \to +\infty)$

We get a L^∞-convergence to zero outside the limit of the x_α's, but the L^2-norm of the gradient is preserved. A bubble might not affect directly some lines in the equation, e.g. when $\Lambda = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

III. The 3-dimensional Kirchhoff system.

Let (M^n, g) be a closed Riemannian n-dimensional manifold, $n = 3$, $p \geq 1$ be an integer, $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ be two sequences of positive real numbers, and $(A_\alpha)_\alpha$ be a sequence of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$. Consider

$$\left(a_\alpha + b_\alpha \sum_{j=1}^{p} \int_M |\nabla u_j|^2 dv_g\right)\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^\alpha u_j = |U|^{2^\star - 2} u_i \quad (S_\alpha)$$

for all $i = 1, \ldots, p$, where $A_\alpha = (A_{ij}^\alpha)_{i,j=1,\ldots,p}$. A sequence $(U_\alpha)_\alpha$ is said to be a sequence of nonnegative solutions of (S_α) if the components of the U_α's are nonnegative and solve the α-equation (S_α) for any α.

We always assume in the sequel that the a_α's and b_α's converge in $\mathbb{R}$, and that the A_α's converge in C^1. We regard such (S_α)'s as perturbations of (S).

The (S_α)'s have a variational structure. Let H^1 be the Sobolev space of p-maps with components in L^2 with one derivative in L^2. The (S_α)'s come with $I_\alpha : H^1 \to \mathbb{R}$ given by

$$I_\alpha(U) = \frac{a_\alpha}{2}\sum_{i=1}^{p}\int_M |\nabla u_i|^2 dv_g + \frac{b_\alpha}{4}\left(\sum_{i=1}^{p}\int_M |\nabla u_i|^2 dv_g\right)^2 + \frac{1}{2}\sum_{i,j=1}^{p}\int_M A_{ij}^\alpha u_i u_j dv_g - \frac{1}{2^\star}\int_M |U^+|^{2^\star} dv_g ,$$

where $U^+ = (u_1^+, \dots, u_p^+)$. Let $(U_\alpha)_\alpha$ be a sequence of p-maps in H^1. The sequence $(U_\alpha)_\alpha$ is a Palais-Smale sequence for $(I_\alpha)_\alpha$ if :

(i) the $I_\alpha(U_\alpha)$'s are bounded,
(ii) and $I'_\alpha(U_\alpha) \to 0$ in $(H^1)'$ as $\alpha \to +\infty$.

Palais-Smale sequences are bounded in H^1 (Brézis-Nirenberg). Conversely, any sequence of solutions of (S_α) which is bounded in H^1 is a Palais-Smale sequence for $(I_\alpha)_\alpha$.

The H^1-theory for the blow-up (Struwe, 84) applies to Palais-Smale sequences : for any PS-sequence $(U_\alpha)_\alpha$ of nonnegative p-maps, up to passing to a subsequence,

$$U_\alpha = U_\infty + \sum_{i=1}^{k} K_\alpha^{\frac{1}{2^\star - 2}} \mathcal{B}_\alpha^i + \mathcal{R}_\alpha \qquad (H^1\text{Dec}) \ ,$$

where, U_∞ is a p-map in H^1, k is an integer, the $(\mathcal{B}_\alpha^i)_\alpha$'s are bubbles, the $\mathcal{R}_\alpha$'s converge strongly to 0 in H^1, and the K_α's, which come from the nonlocal aspects of the equations, are given by

$$K_\alpha = a_\alpha + b_\alpha \sum_{i=1}^{p} \int_M |\nabla u_{i,\alpha}|^2 dv_g \ ,$$

the $u_{i,\alpha}$'s being the components of the U_α's. The sequence $(U_\alpha)_\alpha$ blows up if $k \geq 1$. Define

$$\mathcal{N}(U_\alpha) = \max\Big\{ k \text{ in } (H^1\text{Dec}) \text{ for subsequences of } (U_\alpha)_\alpha \Big\} \ ,$$

the maximal number of bubbles we can have in H^1-dec. of subsequences of the U_α's.

Notations : • Let S be the sharp constant in the Euclidean Sobolev inequality $S\|u\|_{L^{2^\star}}^2 \le \|\nabla u\|_{L^2}^2$,

$$S = \frac{n(n-2)\omega_n^{2/n}}{4} ,$$

where ω_n is the volume of S^n ($S = \frac{3}{4}\omega_3^{2/3}$ when $n = 3$).

• An operator like $\Delta_g + A$ is coercive if : $\exists C > 0$,

$$\|U\|_{H^1}^2 \le C \int_M \left(|\nabla U|^2 + A(U,U)\right) dv_g$$

for all $U : M \to \mathbb{R}^p$ in H^1.

• A matrix A is cooperative if its of diagonal coefficients are nonnegative ($A_{ij} \ge 0$ for all $i \ne j$).

• Given $\Lambda_g > 0$, the Green's function of $\Delta_g + \Lambda_g$ has a singular part in "$1/r$" plus a regular part. It has positive mass if its regular part is positive on the diagonal of $M \times M$. By the positive mass theorem of Schoen-Yau and Witten, there is such a Λ_g on any closed 3-manifold with positive Yamabe invariant.

Theorem 1 : (H.-Thizy, 2014)

Let (M^3, g) be a closed Riemannian 3-manifold, $p \geq 1$ be an integer, $a, b > 0$ be positive real numbers, and $A : M \to M_s^p(\mathbb{R})$ be a C^1-map. For any sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of positive real numbers converging to a and b, any sequence $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of (S_α), there holds that $\|U_\alpha\|_{H^1} = O(1)$. Moreover, if $\Delta_g + \frac{1}{a}A$ is coercive, $-A$ is cooperative, and the U_α's blow up, then

$$a + bS^{3/2}\sqrt{C}\mathcal{N}(U_\alpha) \leq C \ ,$$

where $S = \frac{3}{4}\omega_3^{2/3}$ is the sharp constant in the Euclidean Sobolev inequality, $\mathcal{N}(U_\alpha)$ is as above, $C > 0$ is such that $A \leq C\Lambda_g \mathrm{Id}_p$ in M in the sense of bilinear forms, where Id_p is the identity $p \times p$ matrix, and $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass.

Rk : There are (H.-Wei, 2012) several examples of (S_α) and (S) for which there exist U_α's with $\mathcal{N}(U_\alpha) \geq 1$ (and even $\mathcal{N}(U_\alpha) \gg 1$).

Corollary 1 : (H.-Thizy, 2014)

Let (M^3, g) be a closed Riemannian 3-manifold, and $\Lambda_g > 0$ be such that $\Delta_g + \Lambda_g$ has positive mass. Let $p \geq 1$ be an integer, $a, b > 0$ be positive real numbers, and $A : M \to M_s^p(\mathbb{R})$ be a C^1-map such that $\Delta_g + \frac{1}{a}A$ is coercive, and $-A$ is cooperative. Assume that

$$A(x) < \left(a + \frac{1}{2}b^2 S^3 + \frac{1}{2}bS^{3/2}\sqrt{4a + b^2 S^3} \right) \Lambda_g(x) \mathrm{Id}_p$$

for all $x \in M$, in the sense of bilinear forms, where Id_p is the identity $p \times p$ matrix. Then (S) has a nonnegative nontrivial C^2-solution. Moreover, for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converging to a and b, all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (S_α).

Corollary 2 : (H.-Thizy, 2014)

Let (M^3, g) be a closed Riemannian 3-manifold. Let $p \geq 1$ be an integer, and $A \in M_s^p(\mathbb{R})$ be a positive definite matrix which does not possess nonnegative nontrivial eigenvectors. Then there exists $K \gg 1$ such that for any positive real numbers a and b satisfying that $a + b \geq K$, the Kirchhoff system (S) does not possess nonnegative nontrivial solutions.

Corollary 2 makes sense starting with $p \geq 3$. The matrix

$$A = \frac{1}{42}\begin{pmatrix} 80 & 22 & -26 \\ 22 & 110 & -4 \\ -26 & -4 & 62 \end{pmatrix}$$

is positive definite, has $1, 2, 3$ as eigenvalues, and $(-4, 1, -5)$, $(-1, 1, 1)$, $(-2, -3, 1)$ as corresponding eigenvectors.

By Corollary 2, the assumption in Corollary 1 that $-A$ should be cooperative is necessary.

IV. The n-dimensional Kirchhoff system, $n \geq 4$.

Assume

$$A \equiv \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p \ , \qquad (\star)$$

where S_g scalar curvature of g, and Id_p identity $p \times p$ matrix.

Theorem 2 : (H.-Thizy, 2014)

Let (M^n, g) be a closed Riemannian n-manifold with positive scalar curvature, $n = 4$ or 5, $p \geq 1$ be an integer, $a, b > 0$ be positive real numbers, and $A : M \to M_s^p(\mathbb{R})$ be given by the geometric diagonal model $(\star)$. Assume that

$$\frac{1-a}{b} \notin S^{n/2}\mathbb{N}^\star \ ,$$

where S is the sharp Sobolev constant. For any $\theta \in (0,1)$, there exists $C > 0$ s.t. $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$ converging to a and b, all $(A_\alpha)_\alpha$ in $C^1\left(M, M_s^p(\mathbb{R})\right)$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (S_α).

More results : Let (M^n, g) be a closed Riemannian n-manifold, $n \geq 4$, $p \geq 1$ be an integer, $a, b > 0$, and $A \in C^1\left(M, M_s^p(\mathbb{R})\right)$ be s.t. $\Delta_g + \frac{1}{a}A$ is coercive. Assume one of the following assumptions :

(1) a and b satisfy that $bS^{\frac{n}{2}} a^{\frac{n-4}{2}} > \frac{2}{n-2}\left(\frac{n-4}{n-2}\right)^{\frac{n-4}{2}}$ when $n \geq 5$, and $bS^2 > 1$ when $n = 4$,

(2) (positive geometries) $S_g > 0$ everywhere in M, and

$$A(x) < \frac{(n-2)a}{4(n-1)}\left(1 + bS^{n/2} a^{\frac{n-4}{2}}\right) S_g(x) \mathrm{Id}_p$$

for all $x \in M$, in the sense of bilinear forms,

(3) (nonpositive geometries) $A(x)$ is positive definite for all $x \in M$, $S_g \leq 0$ everywhere in M, and $n = 5$ or $n \geq 7$,

where S_g is the scalar curvature of g, and S is the sharp Sobolev constant. Then for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$ converging to a and b, all $(A_\alpha)_\alpha$ in $C^1\left(M, M_s^p(\mathbb{R})\right)$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (S_α). **+** (if $-A$ is cooperative) Existence of nonnegative (nontrivial) solutions in cases (1) and (2).

Corollary 3 : (H.-Thizy, 2014)

Let (M^n, g) be a closed Riemannian n-manifold, $n \geq 4$. Let $p \geq 1$ be an integer, and $A \in M_s^p(\mathbb{R})$ be a positive definite matrix which does not possess nonnegative nontrivial eigenvectors. Let $\beta, \gamma > 0$ be positive real numbers. Then there exists $K \gg 1$ such that for any positive real numbers a and b satisfying that $a \geq \beta$, $b \geq \gamma$, and $a^{\frac{n-4}{2}} b \geq K$, the Kirchhoff system (S) does not possess nonnegative nontrivial solutions.

Conversely, suppose for instance that $n = 4$, and let v be a solution of $\Delta_g v + hv = v^{2^\star - 1}$. Let a, b, A, and u be such that

$$b \int_M |\nabla v|^2 dv_g < 1 \ , \ \sum_{j=1}^{p} A_{ij} = \left(a + \frac{ab \int_M |\nabla v|^2 dv_g}{1 - b \int_M |\nabla v|^2 dv_g} \right) h \ ,$$

$$u = \sqrt{a + \frac{ab \int_M |\nabla v|^2 dv_g}{1 - b \int_M |\nabla v|^2 dv_g}} \ v \ .$$

Then $U = \left(\frac{1}{\sqrt{p}} u, \ldots, \frac{1}{\sqrt{p}} u \right)$ solves (S).

V. Proof of Theorem 1 (easy part).

Here $n=3$. We want to prove that sequences $(U_\alpha)_\alpha$ of solutions of (S_α) are bounded in H^1 and that the number k of bubbles we can have in H^1-decompositions of such sequences is bounded from above by

$$a + bS^{3/2}\sqrt{C}k \leq C \ ,$$

where $C>0$ is such that $A \leq C\Lambda_g \mathrm{Id}_p$ in the sense of bilinear forms, and $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass. The proof is typical of a 3-dimensional blow-up analysis (close to the one originally developed by Schoen for the Yamabe equation in 3-space dimensions). Let $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ be sequences of positive real numbers, and $(A_\alpha)_\alpha$ be a sequence of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ such that

$$a_\alpha \to a, b_\alpha \to b \text{ as } \alpha \to +\infty \ ,$$

$$A_\alpha \to A \text{ in } C^1 \text{ as } \alpha \to +\infty \ .$$

Also we let $(U_\alpha)_\alpha$ be a sequence of nonnegative nontrivial solutions of (S_α).

We use the 3-dimensional blow-up machinery and get that

$$\begin{aligned} 3-\text{dim. blow-up machinery} &\Rightarrow \text{ Blow-up points are isolated} \\ &\Rightarrow \text{ the } U_\alpha\text{'s are bounded in } H^1. \end{aligned}$$

Then we can assume that $K_\alpha \to K_\infty$ as $\alpha \to +\infty$. Still by the 3-dimensional blow-up analysis we get there need to be a point where the mass of the vectorial Schrödinger operator $K_\infty \Delta_g + A$ is nonpositive. By comparison principles, since $-A$ is cooperative, this implies that $\frac{1}{K_\infty} A$ can't be less than Λ_g. In other words, we use again the 3-dimensional blow-up machinery, and get that

$$\begin{aligned} 3-\text{dim. blow-up machinery} \Rightarrow\ & \exists x \in M, \text{ and } \exists X \in S^{p-1} \text{ s.t.} \\ & A(x).(X,X) \geq K_\infty \Lambda_g(x)\ . \\ & + \dots\ U_\infty \equiv 0. \end{aligned}$$

We recover the H^1-decomposition of the U_α's since the U_α's are bounded in H^1.

In 3-space dimension,

$$\int_M |\nabla \mathcal{B}_\alpha|^2 dv_g = S^{3/2} + o(1) \ .$$

By the splitting of the energy associated with Struwe's (H^1Dec),

$$\begin{aligned} K_\alpha &\stackrel{def}{=} a_\alpha + b_\alpha \int_M |\nabla U_\alpha|^2 dv_g \\ &= a_\alpha + b_\alpha \left(\sqrt{K_\alpha} k S^{3/2} + o(1) \right) \ . \end{aligned}$$

Passing to the limit $\alpha \to +\infty$, $K_\infty = a + bk\sqrt{K_\infty} S^{3/2}$, and then

$$\sqrt{K_\infty} = \frac{bkS^{3/2} + \sqrt{b^2k^2S^3 + 4a}}{2} \ .$$

There exist (see preceding slide) $x \in M$ and $X \in S^{p-1}$ such that $A(x).(X,X) \geq K_\infty \Lambda_g(x)$. By assumption $A \leq C\Lambda_g \mathrm{Id}_p$ in the sense of bilinear forms. Then $K_\infty \leq C$, and we easily get that

$$a + bS^{3/2}\sqrt{C}k \leq C \ .$$

This is exactly what Theorem 1 says. ■

VI. Proof of Corollary 1.

We want to prove that if $A : M \to M_s^p(\mathbb{R})$ is such that $\Delta_g + \frac{1}{a} A$ is coercive, $-A$ is cooperative, and

$$A(x) < \left(a + \frac{1}{2} b^2 S^3 + \frac{1}{2} b S^{3/2} \sqrt{4a + b^2 S^3} \right) \Lambda_g(x) \mathrm{Id}_p$$

for all $x \in M$, in the sense of bilinear forms, where $\Lambda_g > 0$ is a positive function such that $\Delta_g + \Lambda_g$ has positive mass, then

(i) the Kirchhoff system (S) has a nonnegative nontrivial C^2-solution,

(ii) $\forall \theta \in (0,1)$, $\exists C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converging to a and b, all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (S_α).

The proof of (i) and (ii) is based on Theorem 1 showing that Theorem 1 remains valid if we replace the $2^\star$-exponent in (S_α) by subcritical exponents $p_\alpha \leq 2^\star$, $p_\alpha \to 2^\star$.

We consider

$$\Big(a_\alpha + b_\alpha \sum_{j=1}^p \int_M |\nabla u_j|^2 dv_g\Big)\Delta_g u_i + \sum_{j=1}^p A_{ij}^\alpha u_j = |U|^{p_\alpha - 2} u_i \ , \ (\tilde{S}_\alpha)$$

where $a_\alpha \to a$, $b_\alpha \to b$, $A_\alpha \to A$ in C^1, and $p_\alpha \le 2^\star$, $p_\alpha \to 2^\star$ as $\alpha \to +\infty$. Theorem 1 remains true : for any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of $(\tilde{S}_\alpha)$, the U_α's are bounded in H^1 and, up to a subsequence, the number k of bubbles they can have in their H^1-decomposition is s.t. $a + bS^{3/2}\sqrt{C}k \le C$, where $C > 0$ is such that $A \le C\Lambda_g \mathrm{Id}_p$ in the sense of bilinear forms. The subcritical equations always have nonnegative nontrivial solutions (variational arguments). By elliptic theory, it remains to prove that we can't have $k \ge 1$. Corollary 1 holds true if $a + bS^{3/2}\sqrt{C} > C$, and

$$a + bS^{3/2}\sqrt{C} > C \ \Leftrightarrow$$

$$C < a + \frac{1}{2}b^2S^3 + \frac{1}{2}bS^{3/2}\sqrt{4a + b^2S^3} \ .$$

The coercivity of $\Delta_g + \frac{1}{a}A$ implies that the limit profile $U_\infty \not\equiv 0$. This proves Corollary 1. ∎

VII. Proof of Theorem 2 (easy part).

We mix here two type of blow-up arguments. One is the n-dimensional extension of the 3-dim. blow-up argument used to prove Theorem 1. The other one comes from the C^0-theory analysis for blow-up. Let $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$, $(A_\alpha)_\alpha$ be such that $a_\alpha \to a$, $b_\alpha \to b$, $A_\alpha \to A$ in C^1 as $\alpha \to +\infty$. Let $(U_\alpha)_\alpha$ be a sequence of nonnegative nontrivial solutions of (S_α). Then,

(Arg.1) $n-$ dim. extension of the 3-dim. blow-up theory (bounded stability) $\Rightarrow$ the U_α's are bounded in H^1.

Then we get H^1-decompositions for the U_α's, and

(Arg.2) C^0- blow-up theory $+$ Analytic Stability $\Rightarrow$ if $k \geq 1$ then $U_\infty \equiv 0$ (n$=4,5$) $+ \exists x \in M, \exists X \in S_+^{p-1}$ s.t.

$$\left\langle \left(A(x) - \frac{n-2}{4(n-1)} K_\infty S_g(x) \mathrm{Id}_p \right)(X), X \right\rangle_{\mathbb{R}^p} = 0 ,$$

where K_∞ is the limit of the K_α's defined as before.

By assumption

$$A \equiv \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p$$

and thus (by the Analytic Stability theory) we need to have that $K_\infty = 1$. Struwe's H^1-decomposition (and $U_\infty \equiv 0$) implies that

$$\begin{aligned} K_\alpha &\stackrel{def}{=} a_\alpha + b_\alpha \int_M |\nabla U_\alpha|^2 dv_g \\ &= a + bkS^{n/2} K_\alpha^{\frac{2}{2^\star - 2}} + o(1) \ . \end{aligned}$$

Then

$$K_\infty = a + bkS^{n/2} K_\infty^{\frac{2}{2^\star - 2}} \ ,$$

and if $K_\infty = 1$, this implies that

$$\frac{1-a}{b} = S^{n/2} k \ .$$

In other words, $\frac{1-a}{b} \in S^{n/2}\mathbb{N}^\star$ if the U_α's blow up. This proves Theorem 2. ∎

VIII. The red boxes.

We discuss the red lines in the above proofs. There are two results which stand behind these red lines. The first one is attached to the notion of *bounded stability* and has its origin in the 3-dimensional proof of the Yamabe compactness (Schoen, Li-Zhu). The second one is attached to the notion of *analytic stability* and goes back to the C^0-theory by Druet, H., and Robert. Consider

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij} u_j = |U|^{2^\star - 2} u_i \qquad (S_{mod})$$

and perturbations of this system like

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^{\alpha} u_j = |U|^{p_\alpha - 2} u_i \ , \qquad (S_{1,mod}^{\alpha})$$

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^{\alpha} u_j = |U|^{2^\star - 2} u_i \ , \qquad (S_{2,mod}^{\alpha})$$

where $p_\alpha \leq 2^\star$, $p_\alpha \to 2^\star$, and $A_\alpha \to A$ in $C^1\left(M, M_s^p(\mathbb{R})\right)$.

Theorem B : (H.-Thizy, ($S^\alpha_{1,mod}$). Druet-H.-Vétois, 2010, ($S^\alpha_{2,mod}$).)

Let (M^n, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying that

$$A < \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p \tag{1}$$

in M in the sense of bilinear forms. When $n = 3$ assume also that $\Delta_g + A$ is coercive and that $-A$ is cooperative. Then, for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, all sequences $(p_\alpha)_\alpha$ of subcritical/critical powers converging to $2^\star$, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of $(S^\alpha_{1,mod})$. In particular, for any sequence $(A_\alpha)_\alpha$ of C^1-maps converging C^1 to A, any sequence $(p_\alpha)_\alpha$ of subcritical powers converging to $2^\star$, and any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of $(S^\alpha_{1,mod})$, a subsequence of the U_α's converge C^2 to a solution U_∞ of (S_{mod}).

The blow-up analysis behind this theorem is a one bubble analysis. First we prove that blow-up points are isolated, then that we have a finite number of isolated bubbles carrying minimal energy. We conclude with minimum Aubin-Schoen energy type arguments. We eliminate the existence of bubbles thanks to the positive mass theorem when $n = 3$ (Schoen type conclusion), or when the Potential $<$ Geometric Potential (Aubin type conclusion) when $n \geq 4$.

In particular : the potentials in this analysis need to be "small". This is the meaning of (1) in Theorem B.

Theorem B when $p = 1$ goes back to Schoen, Li-Zhu ($n = 3$) and Druet ($n \geq 4$).

The equality case in (1) (Yamabe equation) has been fully adressed and solved in a series of papers by Khuri-Marques-Schoen and Brendle-Marques (surprising unexpected dimensional answer).

The second kind of blow-up approach (the analytic stability approach) is a multi bubble analysis. It starts with the C^0-theory for blow-up (Druet-H.-Robert, 2004), building on the H^1-theory (Struwe, 84). It allows large potentials.

There is a price to pay : the approach deals with sequences of solutions which are bounded in H^1, and it is restricted to $(S^{\alpha}_{2,mod})$-perturbations. These restrictions are necessary :

(i) (Chen-Wei-Yan, 2011) for any $\lambda > \frac{n(n-2)}{4}$ there exists in S^n, $n \geq 5$, sequences $(u_\alpha)_\alpha$ of positive smooth solutions of $\Delta_g u_\alpha + \lambda u_\alpha = u_\alpha^{2^\star - 1}$ such that $\|u_\alpha\|_{H^1} \to +\infty$.

(ii) (Micheletti-Pistoia-Vétois, 2009) On any closed n-manifold, $n \geq 4$, there are potentials $h > \frac{n-2}{4(n-1)} S_g$ for which the equations $\Delta_g u_\alpha + h u_\alpha = u_\alpha^{2^\star - 1 - \varepsilon_\alpha}$ possess blowing-up positive solutions u_α with $\varepsilon_\alpha \to 0$.

We easily pass from (i) and (ii) to the vectorial setting with the opposite condition $A > \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p$.

Theorem C : (Druet-H., 2009. Simplified form.)

Let (M^n, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$ (and $n \neq 6$), $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying that

(H) $\Delta_g + A$ is coercive,

(H') $\forall x \in M$, $A(x) - \frac{n-2}{4(n-1)} S_g(x) \mathrm{Id}_p$ does not possess isotropic vectors.

Then, for any $\theta \in (0,1)$, and any $\Lambda > 0$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of $(S_{2,mod}^\alpha)$ such that $\|U_\alpha\|_{H^1} \leq \Lambda$ for all α. In particular, for any sequence $(A_\alpha)_\alpha$ of C^1-maps converging C^1 to A, and any H^1-bounded sequence $(U_\alpha)_\alpha$ of nonnegative solutions of $(S_{2,mod}^\alpha)$, a subsequence of the U_α's converge C^2 to a solution U_∞ of (S_{mod}).

The point with Theorem C is that we can deal with large A's ! The theorem accepts the opposite inequality $A > \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p$.

The blow-up analysis behind Theorem C is a multi-bubble analysis. We start with the H^1-decomposition and get immediately all blow-up points. The C^0-theory makes that we control in the pointwise sense our sequence $(U_\alpha)_\alpha$ of solutions in terms of the leading H^1-terms (like if $\mathcal{R}_\alpha \equiv 0$). The interaction of bubbles is controlled by the notion of the range of influence of bubbles (Druet). We conclude using an exterior Pohozaev identity.

The main issue with the Kirchhoff equation in high dim is that sequences of solutions of the (S_α)'s are automatically bounded in H^1 when $n \geq 5$, and by Theorem B, they are also automatically bounded in H^1 when $n = 4$ and $S_g > 0$. We can play with both the red boxes Theorem B and Theorem C and get Theorem B type results by using (the more precise) Theorem C (compare Theorem 2 w.r.t. Theorem 1).

Thank you for your attention !